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Abstract
Many empirical size distributions in economics and elsewhere follow Zipf’s law. Starting from the Gibrat assumption, it is essential to add a *second* assumption to explain this phenomenon.

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Many empirical size distributions in economics and elsewhere follow Zipf's law, that is: they exhibit a Pareto distribution - also called Zipf distribution or power law - with an exponent near 1. In economics well-known examples are the distributions of firm sizes, incomes, and city sizes.[1] [2]
This intriguing phenomenon continues to fascinate researchers as the many alternative explanations offered in the literature show. Many of them start from assuming a proportional growth process (the Gibrat assumption) together with a *second* assumption. For example, in Simon (1955) the second assumption is the presence of a constant rate of small new cities, Levy and Solomon (1996) assume a minimum size below which cities cannot decline, Reed (2001) assumes that the time of observation is distributed exponentially[3], while recently Fujiwara et al. (2004) introduce the law of detailed balance. See for an overview De Wit (2005).

Till now, the often cited explanation of Gabaix (1999) seems by far the most elegant of all these explanations, because he claims that in his approach in effect *only* a proportional growth process suffices to arrive at Zipf's law. Unfortunately, this claim – and three other related claims - of Gabaix appear to be incorrect, as I will show in this letter. Hence, the conclusion must be that it is still in debate what the most satisfactory explanation of Zipf's law is.

Gabaix considers a fixed number of cities exhibiting proportional growth. Furthermore, there is a minimum size (s_{min}) below which cities cannot decline. Gabaix (pp. 749-750) proofs that these assumptions lead the city size distribution to converge to a Pareto distribution with an exponent ζ equal to:

$$\zeta = \frac{1}{1-c} \tag{1}$$

where c is defined by s_{min}/s_{av} and s_{av} denotes average city size. Hence, he concludes that "as the minimum city size becomes lower (s_{min} tends to 0, hence becomes almost invisible), the Pareto exponent converges to 1. So, in essence, all we need is an infinitesimally small minimum city size, to ensure that the steady state distribution will be Pareto with an exponent very close to 1." Because the required minimum city size may be infinitesimally small, for all practical purposes it is negligible so that the main message of Gabaix (pp. 739, 741, 760) becomes that "some proportional growth process among cities automatically leads their distribution to converge to Zipf's law".

[1] For presentational reasons I will refer to city sizes in the remainder of this letter.

[2] Zipf's law as an established empirical regulary is not wholly undisputed. See e.g. De Wit (2005) for firm sizes and Eeckhout (2004) for city sizes.

[3] Reed is not aware of the fact that Steindl (1965, p. 53) already assumed this, albeit in a slightly less general context.

Unfortunately, Gabaix's proof contains a flaw.[4] Removing this flaw leads to the following expression defining the Pareto exponent ζ implicitly:

$$N = \frac{\zeta - 1}{\zeta}\left[\frac{(c/N)^{\zeta} - 1}{(c/N)^{\zeta} - (c/N)}\right] \qquad (2)$$

where *N* denotes the total number of cities. This equation is not new: see Malcai et al. (1999). The fact that Gabaix found the incorrect equation (1) instead of (2), has far reaching consequences for the rest of his paper. In fact, it led Gabaix to make four claims that now appear unjustified.

First, what is the true condition for the minimum city size to arrive at a Pareto exponent near 1? As is clear from equation (2), this condition will be dependent on the number of cities *N* in the sample. If we take, as an example, the sample of the 135 American metropolitan areas that Gabaix uses in his paper, one finds from equation (2) that in order to get the Pareto exponent near 1 – say in the interval [0.95, 1.05] - s_{min}/s_{av} should be in the interval [0.13, 0.17].[5]
This gives a completely other picture of the required minimum city size: a negligible minimum city size appears *not* to suffice to ensure a Pareto distribution with an exponent near 1. On the contrary, if the minimum city size becomes too low, the Pareto exponent will become substantially smaller than 1. In the limit it even converges to 0 instead of 1! Hence, the conclusion that a negligible small minimum city size – not worth mentioning - suffices to get Zipf's law should be revised. Instead, it should read that there is a *second* – rather specific - condition necessary for the minimum city size (loosely speaking: it should be small but not too small and certainly not negligible) to explain Zipf's law.
This also explains the puzzling result of Gabaix (p. 749) that only the introduction of an *infinitesimal* small minimum city size is sufficient to change the steady state distribution from a lognormal distribution with ever increasing variance (the standard result of Gibrat, 1931) to a Pareto distribution with an exponent of 1. This result is not true! With the correct equation (2) for the Pareto exponent, we now find that the introduction of an infinitesimal small minimum city size makes the steady state distribution to converge to a Pareto distribution with exponent 0 (instead of 1, which Gabaix finds from equation (1)). A steady-state distribution converging to a Pareto distribution with an exponent of 0 indeed corresponds with a steady-state distribution converging to a lognormal distribution with ever increasing variance, as one would expect. See, e.g., De Wit (2005).

Second, starting point of Gabaix's analysis is a situation in which no new cities are created. An interesting robustness conclusion of the paper is that re-

[4] Gabaix overlooks the fact that in his framework city sizes are normalized, so that the maximum city size is 1 instead of infinity. Hence, when using the normalization condition at the end of the proof, Gabaix should have set the upper bound of city sizes equal to 1 instead of infinity.

5 To get an idea of the influence of the sample size: for a much larger sample - say 10.000 cities - one finds that s_{min}/s_{av} should be in the interval [0.06, 0.11] in order to get the Pareto exponent in the interval [0.95, 1.05].

sults are not affected if the appearance of new cities is introduced, as long as the appearance rate is not too large. This would be important news: the arrived results would be valid in a far greater domain. Besides, it would disprove the result of Steindl (1965) that the appearance rate of new cities has in all circumstances a direct influence on the exponent of the Zipf distribution.

Gabaix's robustness result is based on the following reasoning (pp. 762-763). Gabaix shows that - as long as the appearance rate of new cities is smaller than or equal to the growth rate of existing cities – older age cohorts will eventually always dominate newer age cohorts completely. Since older age cohorts have the time to converge to Zipf's law the total population will also converge to Zipf's law.

However, this reasoning is incorrect. For, by assumption, the average growth rate of each age cohort is positive.[6] Hence, parameter *c* – by definition equal to s_{min}/s_{av} – decreases to zero as time goes on. It follows from the correct equation (2) that in that case the Pareto exponent itself also decreases to zero. In particular, older age cohorts will never converge to Zipf's law as Gabaix finds from the incorrect equation (1).[7] Thus, Steindl's (1965) result is incorrectly challenged by Gabaix.

Third, Gabaix stresses the lack of convergence of the rivalling Simon model when the appearance rate of new cities becomes small. Simultaneously, he claims that there are no convergence problems in his own approach (p. 745). However, also in his approach convergence to the steady-state distribution becomes increasingly troublesome the smaller the minimum city size is chosen.[8] In the limit with an infinitesimally small minimum city size, there would be no convergence altogether! This potential convergence problem in the Gabaix approach can be solved when an extra condition on the existence of a *sizable* minimum city size is introduced.

Fourth, when comparing his approach to those of others, Gabaix (p. 741 and pp. 754-755) stresses that other approaches leading to a Pareto distribution "stopped short of explaining why the Pareto exponent should be 1", while his approach only needs the proportional growth assumption to arrive at a Pareto distribution with an exponent of 1. This alleged advantage of the Gabaix approach no longer holds. We now see that also in the Gabaix approach a rather specific condition on the minimum city size is needed to arrive at a Pareto distribution with an exponent near 1.

In summary, there are many different explanations for the appearance of Zipf's law, all starting from a proportional growth process. Including that of Gabaix, they all need a second assumption to do the trick. It remains a mat-

[6] Otherwise it cannot be larger than the appearance rate of new cities.

[7] It is not even possible to "repair" Gabaix's result by introducing a sizable minimum city size, because - for *any* given minimum size - parameter *c* will eventually diminish to zero as time goes on so that the Pareto exponent will eventually always diminish to zero too.

[8] Gabaix did not recognize that because in his Monte-Carlo simulations he does not investigate how the convergence behaviour depends on the minimum city size.

ter of empirics (or of taste if empirics are not conclusive) which of these explanations should be preferred in a specific situation.

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