Firm Size Distributions

An overview of steady-state distributions resulting from firm dynamics models

Gerrit de Wit

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Summary

The static firm size distributions that we observe in practice are the cumulated result of underlying firm dynamics involving entry of new firms and growth, decline, and exits of incumbent firms. In this paper we give an overview of firm size distributions that result as steady states from models differing in the way these firm dynamics are modelled.

Base model: lognormal and Pareto with a parameter equal to 1
The most basic of models - a fixed number of firms growing according to Gibrat’s law - appears to lead to two different firm size distributions: either the lognormal distribution or the Pareto distribution with a parameter equal to 1 (in the literature also referred to as the Zipf distribution).

Generalizations: four overlapping families of distributions
It appears not possible to incorporate all possible extensions to the base model in one general model while still keeping track of the resulting steady-state firm size distribution. Hence, we review generalizations of the base model in four different (partly overlapping) directions:

1. Models involving different entry assumptions. The most general model in this context produces in the steady state the Waring distribution with two parameters: a parameter $\alpha$ - the fraction of growth due to new firms - and a parameter $\eta$ about the way in which sizes of new firms are distributed.

2. Models involving entries of firms of one size, exits and deviations from Gibrat’s law. The most general model in this context produces in the steady state a particular member of the generalized hypergeometric distribution, which is characterized by three parameters: (i) the fraction of growth due to new firms $\alpha$, (ii) a parameter governing the extent in which Gibrat’s law is violated with respect to growth ($c$), and (iii) a parameter shaping the exits of firms ($e$).

3. Models involving the decline of firms. The most general model in this context produces in the steady state an extended Katz distribution, characterized by three parameters: again the two parameters $\alpha$ (the fraction of growth due to new firms) and $c$ (about the extent in which Gibrat’s law is violated with respect to growth), but now as a third parameter a parameter $d$ governing the extent in which Gibrat’s law is violated with respect to decline.

4. Models with a minimum firm size below which firms cannot decline. In the steady state these models all lead to Pareto distributions with one parameter $\rho$, which is related to the minimum firm size but also - if incorporated in the model - to a substantial entry rate of new firms and/or deviations from Gibrat’s law with respect to the mean or variance of the growth rates of firms.

Steady-state firm size distributions: relationships
In figure 1 the relationship has been sketched between the steady-state firm size distributions following from the base model and the families of distributions following from the four reviewed generalizations of the base model. Note that these families contain many well-known distributions as special or limiting cases. In the figure it has been indicated which particular distribution results when setting parameters to particular values. The various distributions have been ordered in the figure in such a way that (i) the more to the top a distribution is situated the more parameters it has and (ii) the more to the right a distribution is situated the faster it converges to zero (or otherwise stated: the
thinner its right tail is). More specific, on a log-log scale the right tail of the probability density function is a straight decreasing line for the five distributions on the left, it is parabolically decreasing for the lognormal in the middle, it is exponentially decreasing for the five distributions right from the middle, while it is still stronger decreasing for the Poisson distribution on the right.

\textit{Firm size distributions without firm dynamical basis}

In practice, still other functional forms are sometimes used to describe and/or fit empirical firm size distributions. Noteworthy are the generalized beta distributions of the first and second kind and the many special and limiting cases of these distributions, for example: the beta distributions of the first/second kind, the (generalized) gamma distribution, the Singh-Maddala distribution, the lognormal distribution, the Weibull distribution, the Fisk distribution, and the exponential distribution. See McDonald (1984). However, as far as we know there are no firm dynamics models leading to these distributions - except, of course, for the lognormal distribution which follows from the base model. Hence, the parameters of these distributions cannot be related to underlying firm dynamics.

\textit{Influence of firms dynamics on the shape of the firm size distribution}

What general tendencies can be deduced with respect to the influence of firm dynamics on the shape of the resulting steady-state firm size distributions? First, the larger the rate at which new - small - firms enter the industry, the faster the steady-state firm size distribution will converge to zero. This tendency is evident from the steady-state distributions derived in this paper. They all converge faster to zero if the parameter $\alpha$ - the fraction of growth due to new firms - increases. Second, the faster small firms grow with respect to large firms, the more concave the steady-state firm size distribution is and the faster it converges to zero. This tendency can be illustrated by some distributions from figure 1 when realizing that the parameter $c$ is a measure of the extent to which small firms grow faster than large firms. For example, starting from the Waring distribution we find for $c=0$ the Yule distribution, which looks like a straight line on a log-log scale. If we set $c=\infty$ we arrive at the geometric distribution, which is concave and converges faster to zero (exponential on a log-log scale). Or, to give another example, starting from the negative binomial distribution, we find for $c=0$ the logarithmic distribution, for $c=1$ the geometric distribution (faster converging to zero than the logarithmic), and for $c=\infty$ the Poisson distribution (faster converging to zero than the geometric).

Third, the faster small firms decline with respect to large firms, the less concave the steady-state firm size distribution is and the slower it converges to zero. This tendency can also be illustrated by some distributions from figure 1 when realizing that the parameter $d$ is a measure of the extent to which small firms decline faster than large firms. For example, starting from the extended Katz distribution we find for $d=0$ the negative binomial distribution, for $d=c$ a generalization of the logarithmic distribution (converging slower to zero than the negative binomial) and for $d=(c+1)/(1-\alpha)$ the Waring distribution (converging slower to zero than the generalization of the logarithmic distribution).

Fourth, the existence of a minimum firm size below which firms cannot decline may alter the slope of the firm size distribution. The precise value of the resulting slope depends on other particulars: average firm size and the number of firms.

Fifth, the higher the variance of firm growth in a certain size domain, the smaller the slope of the steady-state distribution function in that domain.
Figure 1  Distributions that can be derived as steady states of underlying firm dynamics

Distributions marked by an asterisk have been derived from two or more different firm dynamics models. Hence, their parameters can be interpreted in different ways. The more to the top of the figure a distribution is situated the more parameters it has. The more to the left of the figure a distribution is situated the fatter its right tale (or differently stated: the slower it converges to zero). Exception: Pareto and Yule are very alike and have the same convergence behaviour.
Explaining the shape of firm size distributions in practice
The steady-state approach of this paper appears to offer many plausible explanations of the various shapes of firm size distributions that we encounter in practice.

First, the basic shape of firm size distributions in practice - the Pareto distribution with a parameter equal to 1 - can be explained by the most basic of firm dynamics models: a fixed number of firms growing according to Gibrat’s law.

Second, the appearance of a lognormal firm size distribution can also be explained by this most basic model.

Third, various alternative explanations are offered to explain the often observed concavity of firm size distributions in practice. For example, this might be due to (i) small firms growing faster than large firms on average, (ii) small firms having a higher variance of growth rates, or (iii) the declining of firms.

Fourth, the appearance of firm size distributions with a downward slope steeper than the Pareto distribution with a parameter equal to 1 might be due to a (substantial) entry rate of small new firms. Alternatively, it might be due to the existence of a minimum firm size below which firms cannot decline.

Fifth and last, the appearance of firm size distributions with a downward slope less steep than the Pareto distribution with a parameter equal to 1 might be due to small firms having on average a higher growth rate or a higher variance in their growth. Alternatively, such a slope might also be due to the existence of a minimum firm size below which firms cannot decline.

Of course, the steady-state approach cannot give all answers with respect to explaining the shape of firm size distributions encountered in practice, because of the obvious fact that not all industries are in a steady state. However, as is clear from the above, the steady-state approach nevertheless gives many plausible and valuable clues.

No unique relation between size distribution and dynamics
Starting from certain assumptions about firm dynamics there is - if it exists - a unique steady-state distribution. However, things are not the other way round. There is no unique relationship between a given firm size distribution and the underlying firm dynamics model. In other words, it is possible derive the same steady-state firm size distribution starting from different firm dynamics models. As a consequence, the parameters of a given firm size distribution may be differently interpreted depending on which firm dynamics model one assumes to be responsible for it.

Various examples of this phenomenon may be found in the paper. To mention one, the Waring distribution can be derived from three different models with a fixed entry rate. In two of them there is a particular deviation from Gibrat’s law with respect to growth, either accompanied by the assumption of no declining firms or by the assumption that firms decline in a (very) particular way. In the third model Gibrat’s law is not violated but instead firms enter with different sizes. Accordingly, in the former two models the parameters of the Waring distribution are related to the entry rate of firms and the size of the deviation from Gibrat’s law, whereas in the latter model the parameters are related to the entry rate of firms and the way the sizes of entering firms are distributed.

The gist of the above is that if we find a particular firm size distribution in practice we need extra auxiliary information to decide what the right underlying firm dynamics model is. Only after that has been decided it is possible to interpret the parameters of the firm size distribution.
1 Introduction

The static firm size distributions that we observe in practice are the cumulated result of underlying firm dynamics involving entry of new firms and growth, decline, and exits of incumbent firms. In this paper we give an overview of firm size distributions that result as steady states from models differing in the way these firm dynamics are modelled.

Usefulness
What is the use of such an overview? First, it gives insight into what kind of firm dynamics may be underlying specific firm size distributions. Hence, possible interpretations of the parameters of the size distribution in terms of firm dynamics are provided. Second, relationships between different firm size distributions become clear in terms of underlying firm dynamics. Third, it becomes clear in what way different assumptions concerning entry, exit, growth or decline of firms influence the steady-state firm size distribution. Hence, changing firm size distributions over time or differences between industries might be related to changes or differences in the underlying firm dynamics. Fourth, it gives possible candidates which firm size distributions to use for fitting purposes. Fifth, one gets an impression to what extent it is possible to explain the various shapes of firm size distributions that we encounter in practice by the steady-state approach. Sixth - because the precise relationship between firm dynamics and the resulting firm size distribution becomes clear - determinants of firm dynamics can be translated into determinants of firm size distributions and vice versa.

Surplus value with respect to the literature
This paper draws heavily from the seminal work of Ijiri and Simon (1977). In fact, it can be seen as an update of their book. Then, what is the surplus value compared to their book? First, all relevant firm dynamics models present in their book are grouped in a consistent way, together with the resulting steady-state firm size distributions. Furthermore, we interpret some of their models in a different way than they do, confront them with recent insights, elaborate more upon them, or give the firm size distributions the right labels. For example, this paper introduces the Waring distribution and the extended Katz distribution (already present but unnamed in the work of Ijiri and Simon) into this area of research. Second, a number of new more recent models have been added. Noteworthy, (i) the boundary model of Sutton (1997) has been added and set into perspective, (ii) recent insights from analysing stochastic systems, partly published in physical journals, have been added (in particular those of Malcai et al. (1999) and Gabaix (1999)), (iii) two models of our own have been added: one concerning a general deviation from Gibrat’s law and one concerning small firms exiting more often than large firms. Third, the latest empirical insights concerning the shape of firm size distributions are shortly reviewed (noteworthy those of Sutton (1997) and Axtell (2001) and references therein) and it is analysed to which extent the reviewed models are able to explain these.
Limitations
The models in this paper take assumptions regarding entry, exit, growth and decline of firms as the point of departure. Hence, no models are reviewed that explain these firm dynamics from other economic phenomena, such as R&D expenditures, entrepreneurial ability, technology development, and so on.

Of course, the steady-state approach adopted in this paper cannot give all answers with respect to explaining the shape of firm size distributions encountered in practice, because of the obvious fact that not all industries are in a steady state. However, as will become clear below, the steady-state approach nevertheless gives many plausible and valuable clues.

How do firm size distributions look like in practice?
Before we plunge into theory we give a - very brief - overview of the literature on how firm size distributions look like in practice.¹ In particular, we describe how the probability density looks like on a log-log scale.

- Especially for large number of firms, the density is - for a very long size range - well described by a straight line with a downward slope of approximately -2. That is: a Pareto distribution (see section 2) with a parameter around 1 is found.² Only for very small and very large sizes there is a noteworthy deviation from this line. See e.g. Axtell (2001) and references therein.

- Often - on closer inspection - the above-mentioned straight line is in fact somewhat concave. See e.g. Ijiri and Simon (1977).

- It is also reported (see e.g., Gibrat (1931), Hart and Prais (1956), Hall (1987, p. 584) and Stanley et al. (1995)) that the empirical density can be described quite well by a lognormal density function (see section 6.1). That is: a (concave) parabolic density function is found. Note that this is in line with the previous remark.

- For smaller industries it can appear that a Pareto distribution is found with a parameter substantially different from 1, or even that neither the Pareto nor the lognormal describes the empirical density function satisfactorily. See, e.g., Quandt (1966) and Silberman (1967). Hence, Sutton (1997, p. 52) concludes that probably there is no general density function that describes all empirical densities well.

Sornette and Cont (1997, p. 432) note that the lognormal distribution can be mistaken for an apparent Pareto distribution with a parameter that is slowly varying with the range on which firm sizes are measured.³ This may complicate matters when trying to decide empirically whether the Pareto or the lognormal distribution describes the empirical density function best.

Structure of the paper
The structure of the paper is as follows. In section 2 we introduce the most basic firm dynamics model - a fixed number of firms of which growth follows Gibrat’s law - and show to what steady-state firm size distribution it leads. The next four sections general-

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¹ This overview is partly taken from Marsili (2003). In accordance with the findings reviewed below, she finds - for Dutch manufacturing industries - partly support for the Pareto shape, partly for the lognormal shape.

² For future reference we note that not only values (slightly) above 1 but also values (slightly) below 1 are found. See, e.g., Ijiri and Simon (1977, p. 53) and Axtell (2001).

³ This can be understood quite easily by realizing that if the parabolic shape of the lognormal is stretched it can be approximated by successive straight lines.
ize this model in different directions, while at the same time keeping track to which changes these generalizations lead in the steady-state firm size distribution. In section 3 we generalize the model to allow for entry of new firms. In section 4, also exits of incumbent firms and deviations from Gibrat’s law - that is: growth dependent on firm size - are considered. In section 5 we generalize the model by allowing firms to decline, while in section 6 models are considered in which firms cannot decline below a minimum firm size.
2 Fixed number of firms and Gibrat's law

In this section the most basic firm dynamics model - a fixed number of firms growing according to Gibrat's law - is shortly discussed. It appears that the model has no mathematically well defined steady-state solution. Nevertheless, it points to two firm size distributions: the lognormal distribution (section 2.1) and the Pareto distribution with a parameter equal to 1 (section 2.2). Section 2.3 concludes.

2.1 The lognormal distribution

The first firm dynamics model in history that was used to derive a firm size distribution started from two assumptions (Gibrat, 1931):

1. There are a fixed number of firms.
2. Each firm (of any size) faces the same distribution of growth rates. That is: Gibrat's law holds.

It is easy to derive (see e.g. Sornette and Cont, 1997, p. 432) that in this model the firm size distribution converges to the lognormal distribution (see, e.g., Johnson et al., 1994, pp. 207-209, 211-212):

\[
f(x) = \frac{1}{(x-1)\sqrt{2\pi \sigma}} \exp\left(-\frac{1}{2} \left[\ln(x-1) - \zeta\right]^2 \right), \quad x > 1
\]

Mean: \(\mu = 1 + \exp(\zeta + \frac{1}{2} \sigma^2)\)

Obviously, the lognormal distribution is parabolic on a log-log scale. However, in the limit with large values for the variance (sigma squared), it resembles quite well a Pareto distribution with a parameter near 0 (straight line on a log-log scale) for firm sizes that are not too large. See e.g. Sornette and Cont (1997, p. 432).

The above derived lognormal distribution is not a real steady-state solution, for its parameters are not stable but go to infinity when time goes on. See e.g. Sornette and Cont (1997, p. 432).

2.2 The Pareto distribution with a parameter equal to 1

In the remainder of this paper the above base model is extended in various directions. In these extended models it is possible to take the limit back to the original base model described above. Surprisingly, it can be shown (see the end of sections 3.1 and 6.1) that under certain conditions the steady-state firm size solution of the resulting limiting

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1 All firm size distributions in this paper have size 1 as minimum value in order to keep expressions as simple as possible. Generalization to distributions with an arbitrary minimum value is straightforward.
model - which differs only infinitesimally from the original base model - will be not log-normal but Pareto with a parameter equal to 1:

Density: $f(x) = x^{-2}$ $x \geq 1$

Mean: $\mu = \infty$

Note that the mean and higher moments of this distribution are infinite.

For future reference we will elaborate a little bit on the Pareto distribution. First, the Pareto distribution is also referred to in the literature as a power law or scaling distribution. Second, there are three very similar variants of the Pareto distribution: the continuous Pareto distribution and two discrete variants.

The density of the continuous Pareto distribution reads (Johnson et al., 1994, p. 574):

Density: $f(x) = \rho x^{-(1+\rho)}$ $x \geq 1$

Mean: $\mu = \frac{\rho}{\rho - 1}$ if $\rho > 1$

where $\rho > 0$. On a log-log scale its distribution function as well as its density function are straight downward-sloped lines.

One discrete variant of the Pareto distribution reads (see, e.g., Ijiri and Simon 1977, pp. 75-76):

Density: $f(i) = i^{-\rho} - (i + 1)^{-\rho}$ $i = 1, 2, \ldots$

where $\rho > 0$. Its distribution function has the same shape as that of the continuous Pareto. Hence, it is a straight downward-sloped line on a log-log scale.

Another variant of the discrete Pareto distribution reads (see, e.g., Johnson et al., 1992, pp. 465-471):

Density: $f(i) = \frac{i^{-(1+\rho)}}{\zeta(1+\rho)}$ $i = 1, 2, \ldots$

Mean: $\mu = \frac{\zeta(\rho)}{\zeta(1+\rho)}$ if $\rho > 1$

where $\rho > 0$ and $\zeta(\cdot)$ denoting the Riemann zeta function (see, e.g., Johnson et al., 1992, pp. 29-30). This distribution is also referred to as the (Riemann) zeta distribution or Zipf distribution. For this distribution the density function has the same shape as that of the continuous Pareto. Hence, it is a straight downward-sloped line on a log-log scale.

1 The name Zipf distribution is sometimes reserved for a Pareto distribution with parameter $\rho$ equal to 1.
2.3 Conclusion
The base model - a fixed number of firms growing according to Gibrat's law - points to two firm size distributions: the lognormal and the Pareto with a parameter equal to 1. Both are not really steady-state solutions of the base model. The lognormal has the problem that in the limit its parameters become infinite, while the Pareto can only be derived as a limiting steady-state solution of more extended models. Nevertheless the base model serves as a good starting point of this paper. First, because it has very simple assumptions: a fixed number of firms obeying Gibrat's law. Second, because it produces - be it a bit loosely - the basic shape of the firm size distributions encountered in practice: a Pareto distribution with a parameter equal to 1 or a lognormal distribution.
In the remainder of this paper we will generalize this model in different directions, while keeping track of the induced changes in the steady-state firm size distribution. We will see then that deviations from the basic form (the Pareto distribution with a parameter equal to 1) that we encounter in practice - such as Pareto distributions with a parameter not equal to 1 or slightly concave distributions - can be explained by changes in assumptions of the base model concerning the underlying firm dynamics.
3 Entry of firms

In this section we discuss the effects of different entry assumptions. In section 3.1 a constant entry rate of firms of size 1 is introduced, while in section 3.2 we look at the possibility that new firms enter with different sizes. Section 3.3 concludes.

3.1 Entry of firms with the minimum size

The models of Ijiri and Simon (1977) introduce entry of new firms in the base model. More specific, in their most simple model they start from the following assumptions:

1a. There are entries of firms of size 1. On average, a fraction $\alpha$ of the growth of the industry is due to entries.
1b. There are no exits of firms.
2a. The growth rate of incumbent firms is on average independent of firm size. That is, Gibrat’s law holds.
2b. There are no declining firms.

Ijiri and Simon (1977, pp. 28-33) derive that the steady-state distribution of this model is the Yule distribution (see, e.g., Johnson et al., 1992, p. 276).

Density:

$$f(i) = \rho \left( \frac{i-1}{1+\rho} \right)^{i-1} \quad i = 1, 2, \ldots$$

With:

$$\rho = \frac{1}{1-\alpha} > 1$$

Mean:

$$\mu = \frac{\rho}{\rho-1} = \frac{1}{\alpha}$$

where $(x)_i$ denotes an ascending factorial. As parameter $\rho$ is larger than 1 by construction, the mean of the distribution is finite. Variance and higher moments are only finite if the fraction of growth due to new firms, $\alpha$, is higher than 50%. As such values do not occur in practice, we do not pay attention to these characteristics.

The density function of the Yule distribution is strictly decreasing to zero: there are always fewer firms of size $i$ than firms of size $i-1$. To get an impression of its shape it is good to realize that a Yule distribution with parameter $\rho$ can be approximated fairly well - especially for large values of $i$ - by a Pareto distribution with the same parameter $\rho$ (see Ijiri and Simon, 1977, p. 27, pp. 75-76). Hence, on a log-log scale its right tale is a straight line with a downward slope of $-\rho$.

How does the Yule distribution depend on its only parameter $\alpha$, the fraction of growth due to new firms? If this fraction increases, there will become relative more new small firms because all new firms start with size 1. Hence, we will expect in the steady state a

1 We take this size equal to 1 in order to arrive at a firm size distribution with a minimum value of 1. Generalization to an arbitrary minimum value is straightforward.

2 It is defined by: $(x)_i = x (x+1) (x+2) \ldots (x+i-1)$ for $i = 1, 2, \ldots$ and $(x)_0 = 1$. Hence, $(1) = i!$. 

firm size distribution with a steeper downwardly sloped right tail, while average firm size decreases. This appears indeed the case, because the slope of the right tail is equal to \(-1/(1-\alpha)\), while average firm size is equal to \(1/\alpha\).

**Negligible entry rate: back to the base model**

If we let the fraction of growth due to new firms, \(\alpha\), approach zero, we get as a limiting case the model of section 2: a fixed number of firms that grow according to Gibrat’s law. By setting the parameter \(\alpha\) to 0 - which boils down to setting the parameter \(\rho\) of the Yule distribution equal to 1 - we arrive at the following density function:

Density: \[ f(i) = \frac{1}{i(i+1)} \quad i = 1, 2, \ldots \]

Mean: \(\mu = \infty\)

This specific case of the Yule distribution is also a specific case (\(\rho\) equal to 1) of the first variant of the discrete Pareto distribution that we introduced in section 2. Its mean and higher moments are infinite.\(^1\)

### 3.2 Firms enter with different sizes

In the model of the previous section all firms enter with size 1. In this section we generalize this assumption: firms enter with different sizes. The resulting model has been taken from Ijiri and Simon (1977, pp. 78-81) though our interpretation of the model is different from theirs. More specific, model assumptions become:

1a. On average, a fraction \(\alpha\) of the growth of the industry is due to entries. New firms enter with different sizes, which are distributed geometrically. Hence, the chance that a new firm has size \(i\) is equal to: \((1-\eta) \eta^{i-1}\) with \(\eta\) between 0 and 1.

1b. There are no exits of firms.

2a. The growth rate of incumbent firms is on average independent of firm size. That is, Gibrat’s law holds.

2b. There are no declining firms.

In this model the chance of firm entries with a specific size decreases with a fixed percentage from \((1-\eta)\) to zero with climbing firm size. Hence, the average size of entering firms is \(1/(1-\eta)\). Ijiri and Simon show that this model leads to the Waring distribution (see, e.g., Johnson et al., 1992, pp. 278-279)\(^2\), although they do not label it that way:

Density: \[ f(i) = \left(\frac{d-c}{1+d}\right)^{c-1} \frac{(1+c)^i}{(1+d)^i} \quad i = 1, 2, \ldots \]

With: \[ c = \frac{\eta}{(1-\alpha)(1-\eta)} \quad ; \quad d = \frac{1}{(1-\alpha)(1-\eta)} \]

\(^1\) Because of the discrete approach in the model we find a discrete variant of the Pareto distribution.

\(^2\) Our parameters \(c, d\) below have the following relationship with the parameters in Johnson: \(c = a-1\) and \(d = c-1\). Furthermore, our Waring distribution has been shifted to the right with 1 unit.
Mean: \[ \mu = \frac{d}{d-c-1} = \frac{1}{\alpha (1-\eta)} \]

As average firm size was \(1/\alpha\) in the model of the previous section, average firm size is larger by a factor \(1/(1-\eta)\) in the present model. This makes sense because the average size of entering new firms has also risen by this factor.

Finally, note that the model of the present section incorporates the model of the previous section as a special case. The reader can verify this easily by setting \(\eta=0\). For, in this case assumption 1a boils down to firms entering with a fixed size 1, while the Waring distribution boils down to the Yule distribution.

3.3 Conclusion

In this section the entry of new firms was incorporated in the base model. The most general model in this context produces in the steady state the Waring distribution with two parameters: a parameter \(\alpha\) - the fraction of growth due to new firms - and a parameter \(\eta\) about the way in which sizes of new firms are distributed. Setting the latter parameter equal to zero means in effect that new firms only enter at the minimum size 1. Doing this causes the firm size distribution to boil down to the well-known Yule distribution, introduced by Ijiri and Simon in this area of research.

The introduction of the entry of new firms in the base model does not really change the shape of the firm size distribution: the probability density roughly remains a downward straight line on a log-log scale. However, it changes the slope of the distribution: the more - small - firms enter, the steeper the slope of the right tale of the steady-state distribution.

The results in this section give a possible explanation why in practice in some industries the probability density has a downward slope which is steeper than -1: this could be due to the entrance of new firms. However, there are alternative explanations. See section 6. Note also that the results of this section cannot explain distributions with slopes that are less steep than -1. See for models that are capable of explaining this - again - section 6.
4 Exits of firms and deviations from Gibrat’s law

In this section we extend the base model in different directions: exits of firms and deviations from Gibrat’s law are introduced. Common to the models in this section is that (i) there is a positive entry rate and (ii) firms do not decline. Because exits are introduced, the models in this section are able to describe not only growing industries (as the models of section 3) but also stationary industries.

First we will give a sketch in section 4.1 how size distributions are derived in the models of this section, because it is instructive to have a clue about this. Subsequently, we will describe a model with exits that are independent of firm size (section 4.2), two models in which small firms grow faster than large firms (section 4.3), a model in which the growth rate is inversely proportional to size (section 4.4), a model with arbitrary growth of incumbent firms (section 4.5), and a model in which small firms exit more than large ones (section 4.6). Section 4.7 concludes.

4.1 Sketch of derivation of size distributions

Imagine a stationary industry. In a steady state of this industry the number of firms of a particular size, say $i$, should be constant. Hence, the probability of inflow of firms should be equal to the probability of outflow.

In the models of this section inflow is only possible by growth of smaller firms, while outflow is either due to growth to a larger size or due to exit. Hence, we get the following steady state condition:

\[ \text{‘growth from (i-1) to (i)’} = \text{‘growth from (i) to (i+1)’} + \text{‘exit of firms with size (i)’} \]

The probability that a firm of size (i-1) will grow to size i will be proportional to the number of firms with this size, thus to the steady state density $f(i-1)$. Both the probability that a firm of size i will grow larger or exit will be proportional to the number of firms with size i, thus to the steady state density $f(i)$. It follows that the steady state condition prescribes a relationship between $f(i)$ and $f(i-1)$ for all sizes i. Hence, the complete density steady state density $f(i)$ is determined, because we also know that a density function is normalised to 1.

In Annex I it is shown more explicitly how steady-state distributions can be derived from this principle.

4.2 Exits independent of firm size

Ijiri and Simon (1977, p. 36-37) introduce exits of firms but only exits independent of firm size. Hence model assumptions become:

1a. There are entries of firms of size 1. On average, a fraction $\alpha$ of the growth of the industry is due to entries.
1b. The probability that an incumbent firm exits is for all firms the same. Summed exits are such that the industry as a whole is growing or stationary.
2a. The growth rate of incumbent firms is on average independent of firm size. That is, Gibrat’s law holds.
2b. There are no declining firms.
It is prescribed that the industry as a whole is not declining because otherwise a steady state cannot exist. Because exits do not depend on size in this model, these exits do not influence the steady state size distribution. Hence, the size distribution resulting from this model is the same as that of section 3.1: the Yule distribution.

4.3 Small firms grow faster

Sutton (1997) notes that it is a stylised fact from empirical evidence that Gibrat’s law does not hold exactly. He concludes that - given that a firm survives - small firms grow faster than large firms. To do credit to this finding, we can change model assumptions to:

1a. There are entries of firms of size 1. On average, a fraction $\alpha$ of the growth of the industry is due to entries.
1b. The probability that an incumbent firm exits is for all firms the same. Summed exits are such that the industry as a whole is growing or stationary.
2a. The growth rate of incumbent firms of size $i$ is on average proportional to $1 + c/i$.
2b. There are no declining firms.

Hence, the growth rate is gradually decreasing in this model. The larger the parameter $c$ is, the larger the decrease in the growth rate will be. Smaller firms have a significantly larger growth rate with respect to large firms in this model. On the other hand, large firms among each other have approximately the same growth rate because the term $c$ will become negligible for large firms. Hence, for large firms Gibrat’s law holds approximately in this model.

Interestingly, Ijiri and Simon (1977, p. 38) mention this model briefly. However, they present it not as a refinement of their model (they want to hold to Gibrat’s law in that particular paper), but use it only in a sensitivity analysis.

This model leads to the (shifted) Waring distribution which we already encountered in section 3.2:

Density: $f(i) = (d - c)\frac{(1 + c)c^{-1}}{(1 + d)i}$

With: $d = \frac{1 + c}{1 - \alpha}$

Mean: $\mu = \frac{d}{d - c - 1} = 1/\alpha$

Of course, for $c = 0$ we get the Yule distribution as a special case. How does the Waring distribution depend on the extra parameter $c$?

The density function remains strictly decreasing by the introduction of parameter $c$. However, it becomes concave. For, if $c$ increases, the first value of the density function decreases and at the same time the fastness of the decrease to zero becomes less for small values of $i$. This tendency reverses for larger values of $i$.

At first glance remarkably, average firm size does not change by the introduction of parameter $c$. This can be understood as follows. By definition, the birth rate of new firms of size 1 is $\alpha$. Hence, in a steady state the average size of exiting firms will be equal to $1/\alpha$. As in this and the previous model exits are independent of firm size, the average
size of exiting firms is also equal to the sample average $\mu$. Hence, it follows that in both models the sample mean is the same and equal to $1/\alpha$.

Of course, there are other ways than assumption 2a to model that smaller firms have a higher growth rate than large firms. For example, Ijiri and Simon (1977, pp. 76-78) discuss a model in which the growth rate between incumbent firms is distributed according to a negative binomial distribution. They show that such an assumption leads to a Yule distribution with an incomplete beta function. By the same reasoning as above this distribution should have a mean equal to $1/\alpha$ although Ijiri and Simon do not mention this.

While mentioning this other possibility for the sake of completeness, we will not elaborate on it any further. The reason is that we think that this variation is a bit artificial compared to the more straightforward assumption 2a.

4.4 Growth rate inversely proportional to size

The model of the preceding section includes as a limiting case (viz. when the parameter $c$ approaches infinity) the situation that the expected growth rate of firms is inversely proportional to size, that is: all firms have the same chance to grow with the same absolute amount, independent of their size. In that limiting case model assumptions become:

1a. There are entries of firms of size 1. On average, a fraction $\alpha$ of the growth of the industry is due to entries.
1b. The probability that an incumbent firm exits is for all firms the same. Summed exits are such that the industry as a whole is growing or stationary.
2a. The growth rate of incumbent firms is on average inversely proportional to size.
2b. There are no declining firms.

Hence, the growth rate of firms diminishes to zero for large firms in this model. This model is interesting because Sutton (1997) presents it as a boundary model. He claims that this model provides a good description of the least unequal distribution that we are likely to find in practice at the 4- or 5-digit SIC level. This model leads to the (shifted) geometric distribution (see, e.g., Johnson et al., 1992, pp. 201)\(^1\)

Density: \[ f(i) = \alpha (1 - \alpha)^{i-1} \quad i = 1, 2, \ldots \]

Mean: \[ \mu = 1/\alpha \]

As the distribution is a limiting case of the Waring distribution, it shares its common properties: (i) its density is strictly decreasing and (ii) its mean is equal to $1/\alpha$. However, the geometric distribution is smoother than the Waring distribution. Given a particular fraction of growth due to new firms, $\alpha$, its first value is smaller than all possible first values of the Waring distribution and the fastness with which the density function converges to zero is constant.

\(^1\) This can be checked by taking the limit of the parameter $c$ of the Waring distribution to infinity. Just as what was the case for the Waring distribution, the resulting geometric distribution has been shifted to the right with 1 unit.
4.5 Arbitrary distribution of growth

In the previous two sections we have presented models that deviate from Gibrat’s law in a specific way. However, it is possible to derive a steady-state distribution without specifying the growth of incumbent firms at all. Hence, model assumptions become:

1a. There are entries of firms of size 1. On average, a fraction $\alpha$ of the growth of the industry is due to entries.
1b. The probability that an incumbent firm exits is for all firms the same. Summed exits are such that the industry as a whole is growing or stationary.
2a. The growth rate of incumbent firms of size $i$ is on average proportional to $g(i)/i$.
2b. There are no declining firms.

This model leads to the following distribution:

Density: $f(i) = \frac{\prod_{j=1}^{i-1} g(j)}{\prod_{j=1}^{i} (g(j) + \rho)}$ \hspace{1cm} i = 1, 2, …

Mean: $\mu = \frac{1}{\alpha}$

where $\rho$ is an auxiliary parameter, of which the relationship with the fraction of growth due to new firms $\alpha$ can be established once the growth function $g(j)$ is specified. Hence, it appears that the mean $\mu$ of the size distribution is independent of the way growth is distributed among incumbent firms. The reason for this has already been given in section 4.3.

4.6 Small firms exit more often

Another stylised fact noted by Sutton (1997) is that small firms have a higher chance to exit than larger ones. To do credit to this finding we can change the model to:

1a. There are entries of firms of size 1. On average, a fraction $\alpha$ of the growth of the industry is due to entries.
1b. The probability that an incumbent firm of size $i$ exits is proportional to $1 + e/i$.
   Summed exits are such that the industry as a whole is growing or stationary.
2a. The growth rate of incumbent firms of size $i$ is on average proportional to $1 + c/i$.
2b. There are no declining firms.

Hence, the exit rate is decreasing from $1+e$ to 1 in this model. Note that by setting $e=0$ we get back as a special case our ‘old’ assumption that exits are independent of firm size.

This model leads to the following distribution:

Density: $f(i) = A(\rho, c, e) \frac{(1+c)^{i-1}}{\prod_{j=1}^{i} (j + c + \rho \ (1+e/j))}$ \hspace{1cm} i = 1, 2, …
With: \[ \alpha = \left[ 1 + \frac{\mu + c}{A} \right]^{-1} \]

Mean: \[ \mu > 1/\alpha \]

where \( A \) is a normalizing constant which can be found as a function of the auxiliary parameter \( \rho \) and the parameters of the growth and exit functions \( c \) and \( e \) by normalizing the density to unity.

The above generalization of the Waring distribution is a particular member of the (broad) family of generalized hypergeometric distributions (see, e.g. Johnson et al., 1992, pp. 84-91).

In what way does the Waring distribution change because of the introduction of these size-related exits? First, the density function will become smoother as there will be relatively fewer small firms in the steady state. Furthermore, the average firm size will become larger. For, if small firms exit more often, the average size of exiting firms must be smaller than the sample mean. As in a steady state with a birth rate \( \alpha \) of new firms of size 1, the average size of exiting firms must be equal to \( 1/\alpha \), the sample mean must be larger than \( 1/\alpha \).

4.7 Conclusion

Based on an extensive review of the empirical literature Sutton (1997) concludes that (i) given survival, small firms grow faster on average and (ii) small firms exit more often. To do justice to these two stylised effects we incorporated in this section deviations from Gibrat’s law and exits. The most general model that we reviewed produces in the steady state a particular member of the generalized hypergeometric distribution, which is characterized by three parameters: (i) the fraction of growth due to new firms \( \alpha \), (ii) a parameter governing the extent in which Gibrat’s law is violated with respect to growth \( c \), and (iii) a parameter shaping the exits of firms \( e \).

The above mentioned generalized hypergeometric distribution contains other well-known distributions as special or limiting cases:

- the Waring distribution (if exits are independent of size);
- the Yule distribution (if also no deviations from Gibrat’s law are present);
- the Pareto distribution with a parameter equal to 1 (if also there is no entry of firms so that we are back at the base model);
- the geometric distribution (if Gibrat’s law is violated in the sense that expected growth rate is inversely proportional to firm size).

The latter result is interesting because Sutton (1997) presents the model in which growth rates are inversely proportional to firm size, as a boundary model. He claims that the steady-state distribution that follows from this model - the geometric distribution - provides a good description of the least unequal distribution that we are likely to find in practice at the 4- or 5-digit SIC level.

Apart from providing a boundary distribution for the firm size distributions that we encounter in practice, the generalizations in this section have another practical relevance. For, we have found a possible explanation why probability density functions of firm size distributions appear to a bit concave to the origin on a log-log scale in practice. This might be due to a violation of Gibrat’s law. For, we have found that the parameter \( c \) of the Waring distribution (a measure to which extent Gibrat’s law is violated) causes the steady-state distribution to be concave.
In this section we have particular - relatively simple - assumptions regarding the way the growth and exit rate of incumbent firms depend on size. It would have been possible to use other - more sophisticated - assumptions. See Annex I for a general procedure to do this. This would have led to slightly different size distributions with other parameters. In fact, in section 4.3 we mentioned an alternative assumption regarding the growth rate of incumbent firms and in section 4.5 we presented a model with a general growth rate. However, we think that the used assumptions have the merit of simplicity, while still capturing the essence of Sutton’s stylised facts.
5 Firms decline and exit at the minimum firm size

In this section we discuss models in which it is possible that incumbent firms decline. A feature of these models is that firm exits are taken equivalent to firms of size 1 that decline. In other words, exit of a firm is only possible by first declining step by step to size 1 and subsequently leaving the industry. Another feature of these models is that they describe stationary industries. Thus, contrary to the models of the previous two sections, steady states of growing industries are not described by these models.

We will choose for this section the same structure as the previous one. First, we will sketch in section 5.1 how the steady-state distribution is derived from the steady state condition in these models. Subsequently, we discuss successive generalizations of the base model in section 5.2-5.4. Section 5.5 concludes.

5.1 Sketch of derivation of size distributions

Imagine a stationary industry. In a steady state of this industry the number of firms of a particular size, say \(i\), should be constant. Hence, the probability of inflow of firms should be equal to the probability of outflow. A sufficient condition for this to hold is that the growth of smaller firms to size \(i\) is equal to the decline of firms of size \(i\) to smaller firms:

'growth from (i-1) to (i)' = 'decline from (i) to (i-1)'

The probability that a firm of size (i-1) will grow to size \(i\) will be proportional to the number of firms with this size, thus to the steady state density \(f(i-1)\), while the probability that a firm of size \(i\) declines will be proportional to the number of firms with size \(i\), thus to the steady state density \(f(i)\). It follows that the steady state condition prescribes a relationship between \(f(i)\) and \(f(i-1)\) for all sizes \(i\). Hence, the complete density steady state density \(f(i)\) is determined, because we also know that a density function is normalized to 1.

Note that the above steady state condition is fundamentally different from that of the previous section. First, the growth of smaller firms to size \(i\) is not compensated by the growth of firms of size \(i\) to larger sizes but by the decline of firms of size \(i\) to smaller sizes. Second, there are no direct exits by firms of size \(i\) (except for firms of size 1).

5.2 Firms decline

As a start we analyse a model in which Gibrat’s law holds for growth as well as for decline:

1a. There are entries of firms of size 1. On average, a fraction \(\alpha\) of the growth of the industry is due to entries.
1b. Exits are equivalent to firms of size 1 that decline. There are no other exits.
2a. The growth rate of incumbent firms is on average independent of firm size. That is, Gibrat’s law holds for growth.
2b. The rate of decline of incumbent firms is on average independent of firm size. That is, Gibrat’s law holds for decline.
This model has as a steady state the \textit{logarithmic} distribution (see, e.g., Johnson et al., 1992, p. 285):

\begin{align*}
\text{Density: } f(i) = \frac{1}{-\ln(1 - \lambda_i)} \frac{\lambda_i}{i} = \frac{1}{-\ln \alpha} \frac{(1 - \alpha)^i}{i} & \quad i = 1, 2, \ldots \\
\text{With: } \lambda = 1 - \alpha \\
\text{Mean: } \mu = \frac{1}{-\ln(1 - \lambda)} \frac{\lambda}{1 - \lambda} = \frac{1}{-\ln \alpha} \frac{1 - \alpha}{\alpha} 
\end{align*}

The density function is strictly decreasing. Compared to the Yule distribution it converges to zero with climbing size \(i\) faster. It follows that - given a specific fraction of growth due to new firms \(\alpha\) - the average firm size is smaller.\(^1\)

We conclude that the introduction of declining firms in the base model increases the number of small firms so that the average firm size decreases.

### 5.3 Small firms grow faster

As in section 4 we introduce now a deviation of Gibrat’s law so that smaller firms grow faster than larger ones:

1a. There are entries of firms of size 1. On average, a fraction \(\alpha\) of the growth of the industry is due to entries.

1b. Exits are equivalent to firms of size 1 that decline. There are no other exits.

2a. The growth rate of incumbent firms of size \(i\) is on average proportional to \(1 + c/i\).

2b. The rate of decline of incumbent firms is on average independent of firm size. That is, Gibrat’s law holds for decline.

This generalization appears to lead to the (truncated) \textit{negative binomial} distribution (see, e.g., Johnson et al., 1992, p. 225)\(^2\):

\begin{align*}
\text{Density: } f(i) = \frac{c(1 - \lambda_i)^c}{1 - (1 - \lambda)^c} \frac{(1 + c)_{i-1}}{(i)} \lambda_i & \quad i = 1, 2, \ldots \\
\text{With: } \lambda = 1 - \alpha^{1/c} \\
\text{Mean: } \mu = \frac{c \lambda}{(1 - \lambda)(1 - [1 - \lambda]^c)} = \frac{c (1 - \alpha^{1/c})}{\alpha^{1/c} (1 - \alpha^{1/c})} 
\end{align*}

\(^1\)Ijiri and Simon (1977, p. 17) mistakenly remark that things are the other way round.

\(^2\)Our parameters \(\lambda, c\) below have the following relationship with the parameters in Johnson: \(\lambda = P/Q = P/(1+P) = (Q-1)/Q\), \(c\)ek.
Of course, this distribution reduces to the logarithmic distribution in the limiting case $c=0$.\(^1\)

How does the negative binomial distribution change when the extra parameter $c$ increases?

- The first value of the density function decreases.
- For small values of $i$ the fastness with which the density function converges to zero decreases, while for higher values of $i$ it is the other way round. Hence, for very high values of $c$ combined with a small value of $\alpha$, it is possible that the density function goes up before it decreases to zero.
- Average firm size decreases.

The above is a bit abstract. We can illustrate the dependence on the parameter $c$ by evaluating two specific cases ($c=1$ and $c=\infty$), which lead to well-known distributions.

Let us first assume $c=1$ so that the model assumption concerning the growth of incumbent firms becomes:

2a. The growth rate of incumbent firms of size $i$ is on average proportional to $1 + 1/i$.

In this case we get the (shifted) geometric distribution:

Density: \[ f(i) = (1 - \lambda)\lambda^{-1} = \sqrt{\alpha} \left(1 - \sqrt{\alpha}\right)^{-1} \]

With: \[ \lambda = 1 - \sqrt{\alpha} \]

Mean: \[ \mu = \frac{1}{1 - \lambda} = 1/\sqrt{\alpha} \]

If we take the limiting case where $c=\infty$, the model assumption concerning growth becomes:

2a. The growth rate of incumbent firms is on average inversely proportional to size.

As noted by Ijiri and Simon (1977, p. 61) we get the (truncated) Poisson distribution in this case (see, e.g., Johnson et al., 1992, pp. 181-182):\(^2\)

Density: \[ f(i) = \frac{1}{(e^\lambda - 1)} \frac{\lambda^i}{i!} = \frac{\alpha}{1 - \alpha} \frac{(-\ln \alpha)^i}{i!} \]

With: \[ \lambda = -\ln \alpha \]

Mean: \[ \mu = \frac{\lambda}{1 - e^{-\lambda}} = -\ln \alpha \]

\(^1\) The reader can verify this by using: \[ \lim_{c \to 0} \frac{1 - \alpha^{c/(1+c)}}{c} = -\ln \alpha \]

\(^2\) The reader can verify this by using: \[ \lim_{c \to \infty} c(1 - \alpha^{1/(1+c)}) = -\ln \alpha \]
Note that the density function of the Poisson distribution is not strictly decreasing if the fraction of growth due to new firms $\alpha$ is small, contrary to most of the encountered density functions in this paper.

5.4 Small firms decline faster

In the previous section, we have only looked at models in which the rate of decline of firms is independent of firm size. If we relax this condition then we arrive at the model of Ijiri and Simon (1977, p. 61):

1a. There are entries of firms of size 1. On average, a fraction $\alpha$ of the growth of the industry is due to entries.
1b. Exits are equivalent to firms of size 1 that decline. There are no other exits.
2a. The growth rate of incumbent firms of size $i$ is on average proportional to $1 + c/i$.
2b. The rate of decline of incumbent firms of size $i$ is on average proportional to $1 + d/i$.

This model leads to a (shifted) extended Katz distribution (see, e.g., Johnson et al., 1992, pp. 79-80) \(^1\):

\[
\text{Density: } f(i) = A(\lambda, c, d) \frac{(1 + c)_{i-1}}{(1 + d)_i} \lambda^i \quad i = 1, 2, \ldots
\]

With:

\[
\alpha = \left[1 + \frac{\mu + c}{A}\right]^{-1}
\]

where $A$ is a normalizing constant which can be found as a function of the auxiliary parameter $\lambda$ and the parameters of the growth and decline functions $c$ and $d$ by normalizing the density to unity.

Of course, for $d=0$ we get back the negative binomial distribution. We will not analyse the influence of the extra parameter $d$ in general terms here. Instead, we will discuss briefly two cases for specific values of $d$.

First, let us assume that the decline of incumbent firms is distributed in the same way as growth is distributed among incumbent firms, that is $d = c$, or:

2b. The rate of decline of incumbent firms of size $i$ is on average proportional to $1 + c/i$.

By substituting $d = c$ in the expressions above it follows easily that in this model the following generalization of the logarithmic distribution results:

\[
\text{Density: } f(i) = A(\lambda, c) \frac{\lambda^i}{i + c} \quad i = 1, 2, \ldots
\]

With:

\[
\alpha = \left[1 + \frac{\mu + c}{A}\right]^{-1}
\]

\(^1\) Our parameters $\lambda$, $c$, $d$ below have the following relationship with the parameters in Johnson: $\lambda = \beta$, $c = 1 + \alpha/\beta$ and $d = \gamma$. 

30
We arrive at an interesting special case, if we assume that the deviation from Gibrat’s law is larger for the decline of firms than for the growth of firms (that is parameter $d$ is larger than parameter $c$), the parameter $d$ being exactly equal to $(c+1)/(1-\alpha)$:

$$d = \frac{c+1}{1-\alpha}.$$ 

2b. The rate of decline of incumbent firms of size $i$ is on average proportional to $1 + d i$ with $d = (c+1)/(1-\alpha)$.

Although it is a bit artificial to think that such a coincidence would occur in practice, the case is nevertheless interesting because under this assumption the resulting steady-state distribution appears to be the Waring distribution that we already encountered in sections 3.2 and 4.3:

$$f(i) = \frac{(d-c)(1+c)^{i-1}}{(1+d)}, \quad i = 1, 2, \ldots$$

With:

$$d = \frac{1+c}{1-\alpha}$$

Mean:

$$\mu = \frac{d}{d-c-1} = \frac{1}{\alpha}$$

Hence, we conclude that the steady-state properties of a model assuming no declining firms and no exits or exits independent of firm size (that is the model of section 3.3) can always be exactly reproduced by a model in which the decline of firms has a certain specific relationship with respect to the growth of firms. Otherwise stated, the extended Katz distribution of this section includes the Waring distribution as a special case.

### 5.5 Conclusion

In this section we discussed some models in which the possibility of firm decline was introduced. The most general model that we reviewed produces in the steady state an extended Katz distribution, which is characterized by three parameters: the fraction of growth due to new firms $\alpha$, and two parameters governing the extent in which Gibrat’s law is violated, one with respect to growth ($c$), and one with respect to decline ($d$). The extended Katz distribution contains other well-known distributions as special or limiting cases, noteworthy: the negative binomial distribution, the logarithmic distribution, the geometric distribution, the Poisson distribution, the Waring distribution, the Yule distribution, and the Pareto distribution with a parameter equal to 1.

The first four of these distributions have a probability density that is exponentially decreasing on a log-log scale. This is due to the introduction of declining firms in the underlying models. Hence, we arrive at an alternative explanation why we meet concave probability density functions in practice. The previous section suggested that this might be due to deviations from Gibrat’s law. Now we find this might also be due to the declining of firms.

---

1 This follows most easily by assuming $\lambda=1$ and then checking whether this holds. If we assume $\lambda=1$, it follows from annex II that the normalization constant $A$ is equal to: $d-c$. The mean of the distribution can now be calculated and is equal to $1/\alpha$ (see section 3.3). With these values it is easily checked with expression (2.4) of annex I that $\lambda=1$ indeed holds.
6 Firms do not decline below minimum firm size

The basic model of section 2 has no real steady-state solution. In sections 3-5 the introduction of a constant rate of entering firms solved this problem. Another way to introduce a real steady-state solution is to introduce a minimum firm size below which firms can’t decline. Such models are reviewed here. Section 6.1 assumes a fixed number of firms, sections 6.2 and 6.3 introduce the entry of new firms, while sections 6.4 and 6.5 look at deviations from Gibrat’s law. Section 6.6 concludes.

6.1 Firms have a minimum size

Consider the following model:
1. There are a fixed number of firms.
2. In principle, each firm faces the same distribution of growth rates (Gibrat’s law).

However, there is a minimum firm size below which firms can’t decline.

Hence, in this model the (arbitrary) distribution function of firm growth rates is truncated from below so that firm sizes cannot decline below the minimum firm size. Because - generally speaking - this truncation is only important for small firms near the minimum size, average growth will be higher for these small firms in this model, whereas the variance of growth will be smaller. In this respect, Gibrat’s law is violated a little bit.

It can be shown (see e.g., Gabaix (1999) and Malcai et al. (1999)) that nearly any growth rate distribution truncated in this way leads in the steady state to a Pareto distribution (see section 2.2) for firm sizes:

\[
f(x) = \rho (x_{\text{min}})^{\rho} x^{-(1+\rho)} \quad x >= x_{\text{min}}
\]

\[
\mu = \frac{\rho}{\rho - 1} \quad \text{if } \rho > 1
\]

where \( \rho > 0 \). Interestingly and surprisingly, the steady-state distribution appears to be independent of the distribution of the growth firms of the individual firms. The parameter \( \rho \) is only dependent on the minimum firm size \( x_{\text{min}} \), the average firm size, and the total number of firms. See for the exact relationship Malcai et al. (1999, pp. 1300-1301) and Gabaix (1999, p. 750). From this (rather complicated) exact relationship it follows that for parameter values that are encountered in practice - a relatively large number of firms and a minimum firm size that is low with respect to the average firm size - the parameter \( \rho \) is very near 1. As noted in section 1, such values are often encountered in practice. Furthermore - when the values for the minimum firm size \( x_{\text{min}} \), the average firm size, and the total number of firms are chosen accordingly - it appears possible that the parameter \( \rho \) is below 1. Such a parameter value (sometimes encountered in practice) could not be explained by the models reviewed so far.

The formulation of Malcai et al. is the more general. The formulation of Gabaix corresponds with the approximate formula (16) of Malcai et al. for a very large number of firms. Apparently, Gabaix makes implicitly this assumption in his derivation. Furthermore, Gabaix (1999, p. 744) derives \( \rho = 1 \) because in his heuristic derivation he assumes implicitly that firms have a negligible minimum size.
Assumption 2 of the model of this section is (arguably) the simplest and most elegant way to prevent firms from becoming too small and hence to guarantee a steady-state solution with a Pareto distribution with a parameter near 1. However, it is just one of the possibilities. Another possibility would have been that of Kesten. See Gabaix (1999, pp. 750-751 and pp. 761-762) and references therein.

**Negligible minimum firm size: back to the base model**

If we let the minimum firm size below firms cannot decline, approach zero, we get as a limiting case the model of section 2: a fixed number of firms that grow according to Gibrat’s law. Interestingly, there appear to be two possible limiting distribution functions (see Malcai et al., 1999, p 1301). For a fixed finite number of firms the limiting distribution is just the lognormal distribution that we encountered already in section 2.1. However, if we first take the limit of the number of firms to infinity and after that the limit of the minimum firm size to zero, then we get a Pareto distribution with parameter ρ=1 (see section 2.2 for more information on this distribution):¹

\[
\begin{align*}
\text{Density:} & \quad f(x) = x_{\min} \cdot x^{-2} \quad x \geq x_{\min} \\
\text{Mean:} & \quad \mu = \infty
\end{align*}
\]

6.2 **Moderate entry of new firms**

The above model assumes a fixed number of firms. In this section we relax this assumption a bit by allowing that new firms may enter at a moderate rate. The model becomes:

1a. Firms enter at a rate that is not larger than the rate at which the industry grows.
1b. There are no exits.
2. In principle, each firm faces the same distribution of growth rates (Gibrat’s law). However, there is a minimum firm size below which firms can’t decline.

Gabaix (1999, pp. 751-752) shows that in this case the upper tail of the distribution is still Pareto distributed with a parameter ρ near 1 when the minimum firm size is negligible.

6.3 **Substantial entry of new firms**

Gabaix (1999, pp. 751-752) discusses also the case in which firms enter at a substantial rate:

1a. Firms enter at a rate that is larger than the rate at which the industry grows.
1b. There are no exits.
2. In principle, each firm faces the same distribution of growth rates (Gibrat’s law). However, there is a minimum firm size below which firms can’t decline.

¹ Although Gabaix (1999, pp. 749-750) mentions that the lognormal is a degenerate solution of the model, he finds only the Pareto solution when taking the limit of the minimum firm size to zero. Apparently, he implicitly assumes that the number of firms is infinite.
He shows that in this case the steady-state distribution is still Pareto, although its parameter $\rho$ is larger than 1 in this case. This case is fully consistent with the result of the model of section 3.1 because the Pareto distribution and the Yule distribution of section 3.1 are very alike.

6.4 Average growth dependent on firm size

Gabaix (1999, pp. 756-758) investigates the case in which expected growth rates are dependent on firm size:

1. There are a fixed number of firms.
2. In principle, each firm faces the same distribution of growth rates (Gibrat’s law).
However, there is a minimum size below firms do not decline and in some size domain the expected growth rate is larger.

He shows that in the size domain in which the expected growth rate of firms is larger, the distribution function will be Pareto with a parameter $\rho$ that is smaller than 1. This result is comparable with the result of the model of section 4.3 in which also a situation was analysed in which some firms had a larger expected growth rate than others.

6.5 Variance of growth dependent on firm size

Finally, Gabaix (1999, pp. 756-758) investigates the case in which the variance of firm growth is dependent on firm size:

1. There are a fixed number of firms.
2. In principle, each firm faces the same distribution of growth rates (Gibrat’s law).
However, there is a minimum size below firms do not decline and in some size domain the variance of the growth rate is larger.

He shows that in the size domain in which the variance of firm growth is larger, the distribution function will be Pareto with a parameter $\rho$ that is smaller than 1.

6.6 Conclusion

In this section we discussed models with firms that cannot decline below a minimum firm size. All these models produce in the steady state Pareto distributions with one parameter $\rho$, which is related to the minimum firm size, but also - if incorporated in the model - to a substantial entry rate of new firms and/or deviations from Gibrat’s law with respect to the mean or variance of the growth rates of firms.

These models have two practical consequences. First, they give alternative explanations for finding in practice Pareto distributions with a parameter larger than 1. In section 3 we found that such a parameter could be due to the entry of new firms, a fact confirmed by the models in this section. However, now we find that a parameter larger than 1 could also be solely due to the existence of a minimum firm size below which firms can’t decline.

Second, these models also can give explanations for finding in practice Pareto distributions with a parameter smaller than 1. Such a parameter value might be solely due to the existence of a minimum firm size below which firms can’t decline, or - alternatively - due to the fact that in some size domain the mean or variance of the growth rates of firms is larger than average.
Annex I Calculation of steady-state distributions

Models with deviations from Gibrat's law and exits

In these models it is assumed that (i) the fraction of growth due to new firms of size 1 is \( \alpha \), (ii) the probability of firm growth is proportional to a function \( g(i) \), (iii) firms do not decline, and (iv) the probability of exits of incumbent firms is proportional to a function \( x(i) \).

The general case

Assuming without loss of generality a stationary industry, the steady state condition for the steady state density \( f(i) \) reads:

\[ \text{‘growth from (i-1) to (i)’} = \text{‘growth from (i) to (i+1)’} + \text{‘exit of firms with size (i)’} \]

This condition boils down to:

\[
(1-\alpha) \frac{g(i-1)}{N_g} f(i-1) = (1-\alpha) \frac{g(i)}{N_g} f(i) + \alpha \frac{x(i)}{N_x} f(i)
\]

(1.1)

with:

\[
N_g = \sum_{i=1}^{\infty} g(i) f(i) \quad \text{and} \quad N_x = \sum_{i=1}^{\infty} x(i) f(i)
\]

(1.2)

The reasoning behind this is as follows.

- The term on the left ‘growth from (i-1) to (i)’ should be proportional to \( g(i-1) \) by assumption and to the number of firms of size \( i-1 \) and hence to \( f(i-1) \). The factor \( 1-\alpha \) follows from the fact that the growth from all incumbent firms together should add up to \( 1-\alpha \).
- The expression for first term on the right is explained in the same way.
- The second term on the right ‘exit of firms with size (i)’ should be proportional to \( x(i) \) by assumption and to the number of firms of size \( i \) and hence to \( f(i) \). The factor \( \alpha \) follows from the fact that the exits of all incumbent firms together should add up to the birth rate of new firms \( \alpha \).

Rearranging the steady state condition gives the following relationship between \( f(i) \) and \( f(i-1) \):

\[
f(i)/f(i-1) = \frac{g(i-1)}{g(i) + \rho \times x(i)}
\]

(1.3)

where the auxiliary parameter \( \rho \) is defined as:

\[
\rho = \frac{\alpha}{1-\alpha} \frac{N_g}{N_x}
\]

(1.4)

It follows from (1.3) that the steady state density can be expressed as:
\[ f(i) = \frac{A(\rho, g, x)}{g(1) + \rho x(1)} \prod_{j=2}^{i} \frac{g(j-1)}{g(j) + \rho x(j)} \]  

(1.5)

where \( A \) is a normalizing constant, of which the value can be found as a function of the auxiliary parameter \( \rho \) and the parameters of \( g(j) \) and \( x(j) \) by normalizing the density function to unity.

It remains to derive the relationship between the auxiliary parameter \( \rho \) and the birth rate factor \( \alpha \). The most straightforward expression for \( \alpha \) can be derived by using the steady state condition for firms of size 1:

\[ '\text{entry of firms with size } 1' = '\text{growth from } 1 \text{ to } 2' + '\text{exit of firms with size } 1' \]

which is equivalent to:

\[ \alpha = (1 - \alpha) \frac{g(1)}{N_g} f(1) + \alpha \frac{x(1)}{N_x} f(1) \]  

(1.6)

Using (1.5) for \( f(1) \) and subsequently using (1.4) leads to the following expression for \( \alpha \):

\[ \alpha = \left[ 1 + \frac{N_g}{A} \right]^{-1} \]  

(1.7)

Finally, we will derive a relationship which will prove useful to evaluate the mean of the distribution function. Note that the following should hold in a steady state:

1. The number of exits should be equal to the number of entries.
2. The employment loss due to exits should be equal to the combined employment creation due to new firms and the growth of incumbent firms.

It follows that the average size of exiting firms should be equal to \( 1/\alpha \). Hence, we get:

\[ \sum_{i=1}^{\infty} i \frac{x(i)}{N_x} f(i) = \frac{1}{\alpha} \]  

(1.8)

**Exits independent of size: \( x(i)=1 \)**

In the special case in which exits are independent of firm size (that is: \( x(i)=1 \) and \( N_x=1 \)), the normalization of the density function (1.5) can be done without further specifying the growth function \( g(i) \). In Annex II it is shown that the normalization constant \( A \) is equal to \( \rho \) in this case. Moreover, it follows from (1.8) that the mean of the distribution function will be equal to \( 1/\alpha \). So in this particular case we get for the density function and its mean:

\[ f(i) = \frac{\rho}{g(1) + \rho} \prod_{j=2}^{i} \frac{g(j-1)}{g(j) + \rho} \]  

(1.9)

\[ \mu = \frac{1}{\alpha} \]  

(1.10)
Derivation for special cases

With the general expressions derived above, it is now straightforward to derive the steady-state distributions for the special cases of section 3. The procedure is the following.

1. With help of (1.5) the density function is formulated in terms of the auxiliary parameter \( \rho \), the functions \( g(i) \) and \( x(i) \), and an unknown normalizing constant \( A \).
2. The constant \( A \) is derived in the same terms either by explicit normalizing the density to unity or by using (1.9) if applicable.
3. The mean of the distribution is calculated in the same terms.
4. The relationship between the auxiliary parameter \( \rho \) and the birth rate of new firms \( \alpha \) is derived either by (1.7) or (if applicable) by (1.10). In this step frequently use is made of: \( N_g = 1 \) if \( g(i) = 1 \) and \( N_g = \mu \) if \( g(i) = i \).

Stationary models with declining firms

In these models it is assumed that (i) the fraction of growth due to new firms with size 1 is \( \alpha \), (ii) the probability of firm growth is proportional to a function \( g(i) \), (iii) the probability of firm decline is proportional to a function \( d(i) \), and (iv) exits are equivalent to declining firms of size 1.

The general case

The steady state condition for the steady state density \( f(i) \) reads:

\['growth from (i-1) to (i)\' = \'decline from (i) to (i-1)\'

This condition boils down to:

\[(1 - \alpha) \frac{g(i-1)}{N_g} f(i-1) = \frac{d(i)}{N_d} f(i) \quad (2.1)\]

with: \( N_g = \sum_{i=1}^{\infty} g(i) f(i) \) and \( N_d = \sum_{i=1}^{\infty} d(i) f(i) \) (2.2)

The reasoning behind this is as follows.
- The term on the left ‘growth from (i-1) to (i)’ is explained in the same way as in the model without declining firms.
- The term on the right ‘decline from (i) to (i-1)’ should be proportional to \( d(i) \) by assumption and to the number of firms of size \( i \) and hence to \( f(i) \). The term on the right is normalized in such a way that the decline of all incumbent firms together is equivalent to the growth of all incumbent firms plus the birth rate of new firms.

Rearranging the steady state condition gives the following relationship between \( f(i) \) and \( f(i-1) \):

\[\frac{f(i)}{f(i-1)} = \lambda \frac{g(i-1)}{d(i)} \quad (2.3)\]
where the auxiliary parameter $\lambda$ is defined as:

$$\lambda = (1 - \alpha) \frac{N_d}{N_g} \quad (2.4)$$

It follows from (2.3) that the steady state density can be expressed as:

$$f(i) = \frac{A(\lambda, g, d)}{d(1)} \lambda^i \prod_{j=2}^{i} \frac{g(j-1)}{d(j)} \quad (2.5)$$

where $A$ is a normalizing constant, of which the value can be found as a function of the auxiliary parameter $\lambda$ and the parameters of $g(j)$ and $d(j)$ by normalizing the density function to unity.

It remains to derive the relationship between the auxiliary parameter $\lambda$ and the birth rate factor $\alpha$. The most straightforward expression for $\alpha$ can be derived by using the steady state condition for firms of size 1:

'entry of firms with size 1' = 'exit of firms of size 1'

which is equivalent to:

$$\alpha = \frac{d(1)}{N_d} f(1) \quad (2.6)$$

Using (2.5) for $f(1)$ and subsequently using (2.4) leads to the following expression for $\alpha$:

$$\alpha = \left[ 1 + \frac{N_g}{A} \right]^{-1} \quad (2.7)$$

Derivation for special cases

With the general expressions derived above, it is now straightforward to derive the steady-state distributions for the special cases of section 4. The procedure is the following.

1. With help of (2.5) the density function is formulated in terms of the auxiliary parameter, the functions $g(i)$ and $x(i)$, and an unknown normalizing constant $A$.
2. The constant $A$ is derived in the same terms by normalizing the density to unity.
3. The mean of the distribution is calculated in the same terms.
4. The relationship between the auxiliary parameter $\lambda$ and the birth rate of new firms $\alpha$ is derived by (2.7). In this step frequently use is made of: $N_g=1$ if $g(i)=1$ and $N_g=\mu$ if $g(i)=i$. 


Annex II Normalization of a specific case

First, we will proof the following relationship:

\[ \sum_{i=1}^{\infty} \rho x(i) \prod_{j=1}^{j=i} \frac{g(j-1)}{g(j) + \rho x(j)} = g(0) \]

Note that this is a generalization of the familiar relationship:

\[ \sum_{i=1}^{\infty} \rho \left( \frac{1}{1+\rho} \right)^i = 1 \]

**Proof**

\[ \sum_{i=1}^{\infty} \rho x(i) \prod_{j=1}^{j=i} \frac{g(j-1)}{g(j) + \rho x(j)} = \]

\[ \sum_{i=1}^{\infty} \prod_{j=1}^{j=i} \frac{g(j-1)}{g(j) + \rho x(j)} - \sum_{i=1}^{\infty} \prod_{j=1}^{j=i} \frac{g(j-1)}{g(j) + \rho x(j)} = \]

\[ \sum_{i=1}^{\infty} \prod_{j=1}^{j=i} \frac{g(j-1)}{g(j) + \rho x(j)} - \sum_{i=1}^{\infty} \prod_{j=1}^{j=i} \frac{g(j-1)}{g(j) + \rho x(j)} = \]

\[ \sum_{i=1}^{\infty} \prod_{j=1}^{j=i} \frac{g(j-1)}{g(j) + \rho x(j)} - \sum_{i=1}^{\infty} \prod_{j=1}^{j=i} \frac{g(j-1)}{g(j) + \rho x(j)} = \]

\[ g(0) + \sum_{i=2}^{\infty} \prod_{j=1}^{j=i} \frac{g(j-1)}{g(j) + \rho x(j)} - \sum_{i=1}^{\infty} \prod_{j=1}^{j=i} \frac{g(j-1)}{g(j) + \rho x(j)} = g(0) \]

The above can be used for the normalization of the density function (1.5) for the specific case in which \( x(i) = 1 \):

\[ \sum_{i=1}^{\infty} f(i) = 1 \iff \]
\[
\sum_{j=1}^{\infty} \frac{A}{g(j) + \rho} \prod_{j=2}^{l} \frac{g(j-1)}{g(j) + \rho} = 1 \quad \Leftrightarrow \\
A = \rho
\]
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### Recent Research Reports

<table>
<thead>
<tr>
<th>Report Number</th>
<th>Date</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>H200305</td>
<td>11-9-2003</td>
<td>Modelling Entrepreneurship: Unifying the Equilibrium and Entry/Exit Approach</td>
</tr>
<tr>
<td>H200304</td>
<td>26-6-2003</td>
<td>Immigrant entrepreneurship in the Netherlands</td>
</tr>
<tr>
<td>H200303</td>
<td>18-6-2003</td>
<td>Leadership as a determinant of innovative behaviour</td>
</tr>
<tr>
<td>H200301</td>
<td>9-5-2003</td>
<td>Barriers to Entry</td>
</tr>
<tr>
<td>H200211</td>
<td>13-3-2003</td>
<td>KTO 2003</td>
</tr>
<tr>
<td>H200210</td>
<td>28-2-2003</td>
<td>FAMOS 2002</td>
</tr>
<tr>
<td>H200209</td>
<td>25-2-2003</td>
<td>Wat is de ontwikkeling van het aantal ondernemers?</td>
</tr>
<tr>
<td>H200208</td>
<td>3-2-2003</td>
<td>Strategy and small firm performance</td>
</tr>
<tr>
<td>H200207</td>
<td>21-1-2003</td>
<td>Innovation and firm performance</td>
</tr>
<tr>
<td>H200206</td>
<td>12-12-2002</td>
<td>Business ownership and sectoral growth</td>
</tr>
<tr>
<td>H200205</td>
<td>5-12-2002</td>
<td>Entrepreneurial venture performance and initial capital constraints</td>
</tr>
<tr>
<td>H200204</td>
<td>23-10-2002</td>
<td>PRISMA, The Size-Class Module</td>
</tr>
<tr>
<td>H200203</td>
<td>16-9-2002</td>
<td>The Use of the Guttman Scale in Development of a Family Business Index</td>
</tr>
<tr>
<td>H200202</td>
<td>27-8-2002</td>
<td>Post-Materialism as a Cultural Factor Influencing Entrepreneurial Activity across Nations</td>
</tr>
<tr>
<td>H200201</td>
<td>27-8-2002</td>
<td>Gibrat’s Law: Are the Services Different?</td>
</tr>
<tr>
<td>H200111</td>
<td>21-3-2002</td>
<td>Growth patterns of medium-sized, fast-growing firms</td>
</tr>
<tr>
<td>H200110</td>
<td>21-3-2002</td>
<td>MISTRAL</td>
</tr>
<tr>
<td>H200108</td>
<td>4-3-2002</td>
<td>Startup activity and employment growth in regions</td>
</tr>
<tr>
<td>H200107</td>
<td>5-2-2002</td>
<td>Het model Brunet</td>
</tr>
<tr>
<td>H200106</td>
<td>18-1-2002</td>
<td>Precautionary actions within small and medium-sized enterprises</td>
</tr>
<tr>
<td>H200105</td>
<td>15-10-2001</td>
<td>Knowledge spillovers and employment growth in Great Britain</td>
</tr>
<tr>
<td>H200104</td>
<td>4-10-2001</td>
<td>PRISMA 2001, The Kernel</td>
</tr>
<tr>
<td>H200103</td>
<td>13-8-2001</td>
<td>The Emergence of ethnic entrepreneurship: a conceptual framework</td>
</tr>
<tr>
<td>H200102</td>
<td>12-6-2001</td>
<td>Competition and economic performance</td>
</tr>
<tr>
<td>H200101</td>
<td>12-6-2001</td>
<td>Measuring economic effects of stimulating business R&amp;D</td>
</tr>
<tr>
<td>H200013</td>
<td>25-4-2001</td>
<td>Setting up a business in the Netherlands</td>
</tr>
<tr>
<td>H200012</td>
<td>4-4-2001</td>
<td>An eclectic theory of entrepreneurship: policies, institutions and culture</td>
</tr>
<tr>
<td>H200011</td>
<td>27-3-2001</td>
<td>The Effects of transaction costs and human capital on firm size: a simulation model approach</td>
</tr>
<tr>
<td>H200010</td>
<td>28-2-2001</td>
<td>Determinants of innovative ability</td>
</tr>
<tr>
<td>H200009</td>
<td>22-2-2001</td>
<td>Making sense of the New Economy</td>
</tr>
<tr>
<td>H200008</td>
<td>12-2-2001</td>
<td>KTO2000 - een sectormodel naar grootteklasse voor de analyse en prognose van Korte Termijn Ontwikkelingen</td>
</tr>
</tbody>
</table>